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A note on generalized Robertson–Walker space-times

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A generalized Robertson–Walker (GRW) space-time is the generalization of the classical Robertson–Walker space-time. In the present paper, we show that a Ricci simple manifold with vanishing divergence of the conformal curvature tensor admits a proper concircular vector field and it is necessarily a GRW space-time. Further, we show that a stiff matter perfect fluid space-time or a mass-less scalar field with time-like gradient and with divergence-free Weyl tensor are GRW space-times.

Keywords: Generalized Robertson–Walker space-time; time-like concircular vector field; Lorentzian manifolds with divergence free Weyl tensor; perfect fluid space-times; scalar field space-times with time-like gradient.

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1. Introduction

An n-dimensional generalized Robertson–Walker (GRW) space-time, with $n \geq 3$ is a Lorentzian manifold which is a warped product of an open interval I of $\mathbb{R}$ and an $(n-1)$-dimensional Riemannian manifold. GRW space-times have applications in inhomogeneous space-times admitting an isotropic radiation (see [23]).

An n-dimensional ($n \geq 3$) Lorentzian manifold is named generalized Robertson–Walker space-time if the metric takes the local shape:

$$ds^2 = -(dt)^2 + q(t)^2 g^*_{\alpha\beta} dx^\alpha dx^\beta,$$

where $g^*_{\alpha\beta} = g^*_{\alpha\beta}(x^\gamma)$ are functions of x^γ only $(\alpha, \beta, \gamma = 2, 3, \ldots, n)$ and q is a function of t only. If $g^*_{\alpha\beta}$ has dimension three and has constant curvature, the space is a Robertson–Walker space-time.

For more details about GRW space-times among others, we may mention [1, 2, 4, 7, 13, 22–24]. It is well-known that any Robertson–Walker space-time is a perfect fluid space-time [20] and that in $n = 4$ dimensions a GRW space-time is a perfect fluid if and only if it is a Robertson–Walker space-time (see for example [17], Sec. 4).

A vector field u_j on an n-dimensional semi-Riemannian manifold (M, g) is called torse-forming [19, 21, 25, 30] if

$$\nabla_k u_j = \omega_k u_j + f g_{jk}, \tag{1.1}$$

where f is a scalar function and ω_k is a one-form. The same vector is named concircular [29] if the one-form ω_k is a gradient (one or locally a gradient), that is, there exists a scalar function σ defined on a suitable coordinate domain U of M such that $\omega_k = \nabla_k \sigma$ on this set. Let M be a Lorentzian manifold with the Lorentzian metric g of signature $(-, +, +, \cdots, +)$ and ∇ denote the semi-Riemannian connection. In a Lorentzian manifold u_j is a time-like vector field. Concircular vector fields play an important role in the theory of projective and conformal transformations. Also, such vector fields have applications in general theory of relativity.

For general reference of pseudo-Riemannian geometry, we refer to [10, 20].

Let (M, g) be a Lorentzian manifold of dimension $n(n > 3)$ and C^h_{ijk} be the conformal curvature tensor of type $(1, 3)$ on the Lorentzian manifold M, which is given by

$$\begin{aligned} C^h_{ijk} = R^h_{ijk} &- \frac{1}{n-2}[R^h_k g_{ij} - R^h_j g_{ik} + R_{ij}\delta^h_k - R_{ik}\delta^h_j] \\ &+ \frac{R}{(n-1)(n-2)}[g_{ij}\delta^h_k - g_{ik}\delta^h_j], \end{aligned}$$

where R^h_{ijk}, R_{ij} and R denote respectively the curvature tensor, $(0, 2)$ Ricci tensor and scalar curvature.

We assume that in a Lorentzian manifold $(M, g)(n > 3)$, the divergence of the conformal curvature tensor vanishes, that is, $\nabla_m C^m_{ijk} = 0$ and the Ricci tensor R_{ij} satisfies the condition

$$R_{ij} = -R u_i u_j, \tag{1.2}$$

where u_i is a unit time-like vector field. In such a case, the manifold is termed Ricci simple [12]. As a particular case of $\nabla_m C^m_{ijk} = 0$, we may have $C^h_{ijk} = 0$, that is, a conformally flat manifold. For the geometric meaning of a conformally flat Riemannian manifolds, see [31].

The condition (1.2) has a geometric meaning that a unit time-like vector u_i becomes a principal vector of the Ricci operator. Also this condition is time-like partial Einstein manifolds, i.e. it is Einstein with respect to the time-like subbundle. Condition (1.2) plays an important role in general relativity. Let (M, g) be an n-dimensional $(n > 3)$ Lorentzian manifold equipped with Einstein's field equation without cosmologigal constant, that is,

$$R_{ij} - \frac{R}{2} g_{ij} = \kappa T_{ij} \tag{1.3}$$

being $\kappa = \frac{8\pi G}{c^4}$ the Einstein's gravitational constant and T_{ij} the energy–momentum tensor (see [27] and [14]) describing the matter content of the space-time. If the condition (1.2) is fulfilled, then inserting in Einstein's equations, we infer

$$T_{ij} = \frac{2T}{n-2} u_i u_j + \frac{T}{n-2} g_{ij} \tag{1.4}$$

being $T = g^{ij} T_{ij}$ and $R = \frac{2\kappa T}{2-n}$. This is the expression of a perfect fluid energy–momentum tensor (see [18, 27, 14, 28]) $T_{ij} = (\mu + p) u_i u_j + p g_{ij}$, where $\mu = \frac{T}{n-2}$ is the energy density and $p = \frac{T}{n-2}$ is the isotropic pressure and u_j the fluid velocity. Usually in a perfect fluid p and μ are related by an equation of state of the form $p = p(\mu, \Theta)$, being Θ the absolute temperature. In the situation in which the state equation reduces to the form $p = p(\mu)$ the fluid is named isentropic. In our case, $p = \mu$ and the perfect fluid is termed *stiff matter* (see [27], p. 66). Conversely, if an energy–momentum perfect fluid form is specified, then the Ricci tensor is written in the form $R_{ij} = \kappa(\mu+p) u_i u_j + \frac{\kappa g_{ij}}{2-n}(p - \mu)$; in this way a stiff matter model gives rise to a Ricci tensor of the form $R_{ij} = -R u_i u_j$ with $R = -2\kappa\mu$. A stiff matter equation of state was firstly introduced by Zel'dovich in [32] and used by the same author in his cosmological model exposed in [33]; in this paper the primordial universe is assumed to be a cold gas of baryons. For a stiff matter fluid, the sound velocity is equal to the velocity of the light [32]. For recent results on the stiff matter era of the universe see for example [9]. The stiff matter era preceded the radiation era, with $p = \frac{\mu}{3}$, the dust matter era, with $p = 0$ and the dark matter era, with $p = -\mu$ [9]. It also occurs in certain cosmological models where dark matter is made of relativistic self-graviting Bose–Einstein condensate [8].

Energy–momentum tensor of perfect fluid type also arise from scalar field space-times with time-like gradient $\nabla_j \psi$; stiff matter models are recovered from mass-less fields (see [27], p. 63). In fact the energy–momentum of a real spin-0 field ψ is defined by (see [18, 28])

$$T_{ij} = (\nabla_i \psi)(\nabla_j \psi) - \frac{1}{2} g_{ij} [(\nabla_k \psi)(\nabla^k \psi) + V(\psi)], \tag{1.5}$$

where $V(\psi)$ is a potential that models the self-interaction between particles, whose simplest form is $V(\psi) = \frac{m^2}{2\hbar^2}\psi^2$ (m is the particle mass and $\hbar$ is the Planck's constant divided by 2π). In the case of mass-less time-like gradient field setting $u_k = \frac{\nabla_k \psi}{\sqrt{|(\nabla_j \psi)(\nabla^j \psi)|}}$ we have a perfect fluid of the form

$$T_{kl} = |(\nabla_j \psi)(\nabla^j \psi)| u_k u_l + \frac{1}{2} g_{kl} |(\nabla_j \psi)(\nabla^j \psi)| \tag{1.6}$$

with $p = \mu = \frac{1}{2}|(\nabla_j \psi)(\nabla^j \psi)|$.

In a recent paper [11], Chen proved that an n-dimensional Lorentzian manifold with $n \geq 3$ is a generalized Robertson–Walker space-time if and only if it admits a time-like concircular vector field of the form $\nabla_k X_i = \rho g_{ik}$.

The purpose of this note is to characterize a GRW space-time by proving the following theorem:

Theorem 1.1. *Let (M, g) be an n-(> 3) dimensional Lorentzian manifold. If the Ricci tensor has the form $R_{ij} = -R u_i u_j$ and the divergence of the conformal curvature tensor vanishes, that is, $\nabla_m C^m_{jkl} = 0$, then there exists a suitable coordinate domain U of M such that on this set the space is a* GRW *space-time with Einstein fibers.*

In view of the aforementioned considerations about stiff matter models and mass-less scalar fields with time-like gradient $\nabla_j \psi$ we can state the following results.

Corollary 1.1. *Let (M, g) be an n-dimensional $(n > 3)$ perfect fluid space-time subjected to the condition $p = \mu$. If the divergence of the conformal curvature tensor vanishes, then there exists a suitable coordinate domain U of M such that on this set the space is a* GRW *space-time with Einstein fibers.*

Corollary 1.2. *Let (M, g) be an n-dimensional $(n > 3)$ mass-less scalar field space-time with time-like gradient $\nabla_j \psi$. If the divergence of the conformal curvature tensor vanishes, then there exists a suitable coordinate domain U of M such that on this set the space is a* GRW *space-time with Einstein fibers.*

Remark 1.1. In [26] the solutions of Einstein's equations are studied under the following assumptions: (1) the space is a perfect fluid 4-dimensional space-time; (2) the divergence of the conformal curvature tensor vanishes, that is, $\nabla_m C^m_{jkl} = 0$; (3) the space is equipped with a state equation $p = p(\mu)$. It was thus proved that the space-time is conformally flat and the metric is a Robertson–Walker metric. The flow is irrotational, shear free and geodesic. It should be noted that our Theorem 1.1 is proven in any dimensions and without using a state equation.

2. Proof of the Main Theorem

To prove our main theorem, we first prove the following key lemma.

Lemma 2.1. *Let (M, g) be an n-dimensional $(n > 3)$ Lorentzian manifold admitting a unit concircular vector field of the form* (1.1); *then there exists a suitable coordinate domain U of M such that on this set the space is a* GRW *space-time.*

Proof. Let us assume that a Lorentzian manifold is equipped with a unit time-like concircular vector field of the form

$$\nabla_k u_j = f g_{jk} + \omega_k u_j, \tag{2.1}$$

where ω_k is a closed one-form. It should be noted that for unit time-like torseforming or concircular vectors, it is $\omega_k = f u_k$ (see [19]).

Now if ω_k is closed, then there exists a scalar function σ defined on a suitable coordinate domain U of M such that $\omega_k = \nabla_k \sigma$ on this set.

Setting $X_j = u_j e^{-\sigma}$ on this set we have

$$\begin{aligned}\nabla_k X_j &= e^{-\sigma}(\nabla_k u_j - u_j \nabla_k \sigma) \\ &= e^{-\sigma}[(\nabla_k \sigma)u_j + f g_{jk} - u_j(\nabla_k \sigma)] \\ &= (e^{-\sigma} f) g_{jk}\end{aligned}$$

and consequently

$$\nabla_k X_j = \rho g_{jk},$$

where $\rho = e^{-\sigma} f$ is a scalar function and $X_j X^j = -e^{-2\sigma} < 0$ is a time-like vector.

The previous equation can be written as

$$\nabla_k X_j + \nabla_j X_k = 2\rho g_{jk},$$

that is, X_j is a conformal Killing vector. The previous rescaling was suggested in [19].

In a recent paper [11], Chen has shown that the presence of a time-like concircular vector of the form $\nabla_i X_j = \rho g_{ij}$ is equivalent to have a GRW space-time. Then if an n-dimensional ($n > 3$) Lorentzian manifold is equipped with a unit concircular vector field of the form $\nabla_k u_j = f g_{jk} + \omega_k u_j$, then there exists a suitable coordinate domain U of M such that on this set the space is a GRW space-time. □

Proof of Theorem 1.1. Let us assume that (M, g) be a Lorentzian manifold of dimension $n(n > 3)$ with Lorentzian metric g. The divergence of the conformal curvature tensor vanishes, that is, $\nabla_m C^m_{jkl} = 0$ implies

$$\nabla_k R_{jl} - \nabla_l R_{jk} = \frac{1}{2(n-1)}(g_{ij}\nabla_k R - g_{jk}\nabla_l R).$$

Now we write the covariant derivative of (1.2) and insert in the previous equation to obtain

$$\begin{aligned}-(\nabla_k R)u_j u_l - R(\nabla_k u_j)u_l - R(\nabla_k u_l)u_j \\ + (\nabla_l R)u_j u_k + R(\nabla_l u_j)u_k + R(\nabla_l u_k)u_j \\ = \frac{1}{2(n-1)}(g_{ij}\nabla_k R - g_{jk}\nabla_l R).\end{aligned} \tag{2.2}$$

Transvecting (2.2) with g^{jl} we obtain

$$\frac{1}{2}(\nabla_k R) + [(\nabla_l R)u^l + R(\nabla_l u^l)]u_k + R u^l(\nabla_l u_k) = 0. \tag{2.3}$$

On the other hand, transvecting (2.2) with u^j, we have

$$R(\nabla_k u_l - \nabla_l u_k) = \frac{3-2n}{2(n-1)}(u_l \nabla_k R - u_k \nabla_l R). \tag{2.4}$$

Transvecting (2.4) again with u^l we get

$$R(\nabla_l u_k)u^l = \frac{3-2n}{2(n-1)}(\nabla_k R + u_k u^l \nabla_l R).$$

Substituting this in (2.3), we obtain

$$\frac{-n+2}{2(n-1)}(\nabla_k R) + \left[\frac{1}{2(n-1)}(\nabla_l R)u^l + R(\nabla_l u^l)\right] u_k = 0. \tag{2.5}$$

Transvecting this with u^k, we have

$$R(\nabla_k u^k) = -\frac{1}{2}(\nabla_k R)u^k.$$

Inserting this in (2.5), we get

$$\nabla_k R = -(\nabla_l R)u^l u_k. \tag{2.6}$$

So (2.4) becomes

$$R(\nabla_k u_l - \nabla_l u_k) = 0. \tag{2.7}$$

Inserting (2.7) and (2.6) in (2.2) we get

$$R[(\nabla_l u_j)u_k - (\nabla_k u_j)u_l] = -\frac{(\nabla_m R)u^m}{2(n-1)}(g_{jl}u_k - g_{jk}u_l). \tag{2.8}$$

Transvecting (2.7) with u^l we obtain

$$(\nabla_k u_j) = -\frac{(\nabla_m R)u^m}{2R(n-1)}(u_l u_k + g_{kl}). \tag{2.9}$$

We have obtained a unit time-like torseforming vector field with $f = -\frac{(\nabla_m R)u^m}{2R(n-1)}$. We prove that $\omega_k = f u_k$ is a closed one-form. From (2.6) we have

$$\nabla_j \nabla_k R = -(\nabla_j u_k)(\nabla_m R)u^m - u_k \nabla_j((\nabla_m R)u^m). \tag{2.10}$$

Writing an analogous equation with indices j and k exchanged, comparing them we infer $u_k \nabla_j((\nabla_m R)u^m) = u_j \nabla_k((\nabla_m R)u^m)$ and thus

$$\nabla_j((\nabla_m R)u^m) = -u_j u^k \nabla_k((\nabla_m R)u^m). \tag{2.11}$$

Now a covariant derivative of f gives

$$\nabla_j f = -\frac{\nabla_j((\nabla_m R)u^m)}{2R(n-1)} + \frac{(\nabla_m R)u^m}{2(n-1)} R^{-2}(\nabla_j R) \tag{2.12}$$

and in view of the previous results and of (2.6) we infer $\nabla_j f = \eta u_j$ for a suitable scalar function η. In this way, we have $\nabla_j \omega_k = \nabla_k \omega_j$ and u_j is a unit time-like concircular vector.

Hence by Lemma 2.1, we conclude that if in a Lorentzian manifold M, the divergence of the conformal curvature tensor vanishes and the Ricci tensor satisfies (1.2), then there exists a suitable coordinate domain U of M such that on this set the space is a GRW space-time. Following arguments similar to those used by Gębarowski in [15, Lemma 7] it is possible to show that for a metric of the form

$$ds^2 = \varepsilon(dt)^2 + q(t)^2 g^*_{\alpha\beta}(x_2 \cdots x_n) dx^\alpha dx^\beta,$$

where $\alpha, \beta \in \{2, \ldots, n\}$ and $\varepsilon = \pm 1$ the condition $\nabla_m C^m_{jkl} = 0$ is equivalent to $R^*_{\alpha\beta} = \frac{R^*}{n-1} g^*_{\alpha\beta}$. Thus the fibers of the GRW space-times are Einstein (see also [16], Lemma 1). □

Remark 2.1. Let us give a look at the conformally flat case. From Theorem 1(i) of [6], it is well-known that the metric $ds^2 = -(dt)^2 + q(t)^2 g^*_{\alpha\beta} dx^\alpha dx^\beta$ is conformally flat if and only if $g^*_{\alpha\beta}$ is a space of constant curvature. Thus we have the following:

Proposition 2.1. *Let (M, g) be an n-dimensional $(n > 3)$ conformally flat Lorentzian manifold. If the Ricci tensor has the form $R_{ij} = -Ru_i u_j$ then there exists a suitable coordinate domain U of M such that on this set the space is a Robertson–Walker space-time.*

Corollary 2.1. *Let (M, g) be an n-dimensional $(n > 3)$ conformally flat perfect fluid space-time subjected to the condition $p = \mu$; then there exists a suitable coordinate domain U of M such that on this set the space is a Robertson–Walker space-time.*

Corollary 2.2. *Let (M, g) be an n-dimensional $(n > 3)$ conformally flat massless scalar field space-time with time-like gradient $\nabla_j \psi$; then there exists a suitable coordinate domain U of M such that on this set the space is a Robertson–Walker space-time.*

We conclude with the following two remarks.

Remark 2.2. We only mention that Einstein GRW space-times were classified in [3]. In that case, the warping function q is subjected to some restrictions specified by the following differential equations (reported in [5])

$$qq'' = \frac{R}{n(n-1)} q^2, \quad \frac{R}{n(n-1)} q^2 = \frac{R^*}{(n-2)(n-1)} + (q')^2.$$

Setting $q = \sqrt{F}$ it is inferred that

$$FF'' - (F')^2 - \frac{2R^*}{(n-1)(n-2)} F = 0.$$

In [5] it is pointed out that the following functions are solutions of the previous equation

$$F(t) = -C_1 \left(t - \frac{c}{C_1} \right)^2, \quad C_1 < 0,$$

$$F(t) = \frac{c}{2}\left(e^{\pm\frac{d}{2}t} + \frac{2C_1}{d^2c}e^{\mp\frac{d}{2}t}\right)^2, \quad c > 0,\ d \neq 0,$$
$$F(t) = -\frac{2C_1}{c}(1 + \sin(ct + d)), \quad \frac{C_1}{c} < 0 \tag{2.13}$$

being c, d some constants and $C_1 = \frac{R^*}{(n-1)(n-2)}$.

Remark 2.3. To prove his theorem Chen puts $X = \phi e_1$ (see [11], Eq. (3), where e_1 is a unit time-like vector in the direction of X: consequently in the proof he can choose $q = \phi$ for the warping function. In our proof, we have posed $X_j = e^{-\sigma}u_j$ so it is $\phi = e^{-\sigma}$ and we can choose $q = e^{-\sigma}$ for the warping function. But we have $\omega_k = fu_k = \nabla_k\sigma$ so that $f = -u^k\nabla_k\sigma$ and finally $f = u^k\nabla_k(\ln q)$. This is the relation between the warping function q and the function f of the concircular vector.

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